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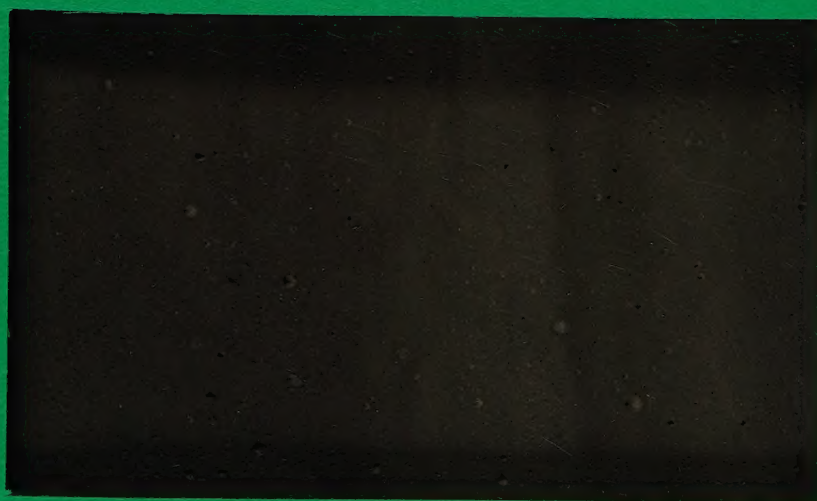
THE JACOBI FORM AND THE
SECOND VARIATION

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The Jacobi form and the second variation

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0. Introduction

In [3] and [4] Peetre gave intrinsic formulas for the Euler equations for n :th order single integral variational problems. The author extended the formula for the so-called Euler derivative in the case of a manifold with a connection to general multiple integral problems (see [1]). This approach to the calculus of variations has proved fruitful in the examples worked out in these papers. The formulas derived may have the defect of being long, but they make for manageable and often very quick calculations.

So in this report, we thought it might be nice to give a version of the second variation using this technique. We treat first order problems, and consider the single integral case to start with. For the second variation, it is natural to introduce the Jacobi form. In the instance of a geodesic on a Riemannian manifold, the Jacobi form expresses precisely the usual Jacobi equation.

Multiple integrals are handled analogously. As an example, we take the second variation of the volume of a minimal surface. It might be appropriate to cite Spivak [5], where a calculation along standard lines is given: "The calculation itself is a real bitch,..."

1. The first variation for first order single integral problems

We begin by recalling the formalism leading to the first variation formula in Peetre [2].

Let f be a C^∞ function on TM , the tangent bundle of a manifold M . Suppose that X and Y are two vector fields defined on the same open subset of M . Then we define the Lie derivative of f with respect to Y , $L_Y f$, by

$$L_Y f(X) = \frac{d}{d\varepsilon} f(\varphi_\varepsilon^* X) \Big|_{\varepsilon=0}.$$

φ_ε denotes the local flow generated by Y .

Let p be a point where the vector fields are defined. Write

$$(\varphi_\varepsilon^* X)_{\varphi_\varepsilon(p)} = \varphi_{\varepsilon^* X} p = \varphi_{\varepsilon^* X} \varphi_{-\varepsilon} \circ \varphi_\varepsilon(p) = (X^{\varphi_{-\varepsilon}})_{\varphi_\varepsilon(p)},$$

where thus $\frac{d}{d\varepsilon} X^{\varphi_{-\varepsilon}} = -[Y, X]$. Letting $\dot{f}$ denote the fiber derivative of f , defined by

$$\dot{f}(U; X) = \frac{d}{d\varepsilon} f(X + \varepsilon U) \Big|_{\varepsilon=0}$$

for a tangent vector U defined over the same point as X , we obtain

$$L_Y f(X) = Yf(X) - \dot{f}([Y, X]; X).$$

Put

$$L_X \dot{f}(Y; X) = X \dot{f}(Y; X) - \dot{f}([X, Y]; X).$$

Then $L_Y f$ can also be written as

$$L_Y f(X) = Yf(X) - L_X \dot{f}(Y; X) + X \dot{f}(Y; X).$$

Defining the Euler derivative $E_Y f$ by

$$E_Y f(X) = L_Y f(X) - X \dot{f}(Y; X) = Yf(X) - L_X \dot{f}(Y; X),$$

we therefore have a form $Y \mapsto E_Y f(X)$.

We apply this formalism to a variational problem in the following way. Let $[a, b] \ni t \mapsto x(t) \in M$ be a C^∞ curve, and set $X = \frac{dx}{dt}$.

Suppose that we are given an infinitesimal deformation Y along the curve. If we extend X and Y in some way, and put

$$J(\varepsilon) = \int_a^b f(\varphi_\varepsilon * X) dt,$$

then

$$\left. \frac{dJ}{d\varepsilon} \right|_{\varepsilon=0} = \int_a^b L_Y f(X) dt = \int_a^b E_Y f(X) dt + [\dot{f}(Y; X)]_a^b.$$

Because the formulas for $E_Y f(X)$ and $\dot{f}(Y; X)$ are independent of the extensions chosen, we have arrived at a bona fide first variation formula.

If M comes equipped with a connection ∇ with torsion T and curvature R , then it is easily verified that

$$L_Y f(X) = \nabla_Y f(X) + \dot{f}(\nabla_X Y) + \dot{f}(T(Y, X))$$

and

$$E_Y f(X) = \nabla_Y f(X) - \nabla_X \dot{f}(Y) + \dot{f}(T(Y, X))$$

where we have defined $\nabla_Y f$ and $\nabla_X \dot{f}$ as follows:

$$\nabla_Y f(X) = Yf(X) - \dot{f}(\nabla_Y X)$$

and

$$\nabla_X \dot{f}(Y) = Xf(Y) - \dot{f}(\nabla_X Y).$$

2. The second variation

Suppose now that we have given a second infinitesimal variation Z along the curve. Repeating the procedure of section 1, we get the integral

$$\int_a^b L_Z L_Y f(X) dt = \int_a^b E_Z L_Y f(X) dt + [\dot{\overline{L}}_Y f(Z)]_a^b.$$

We call $Z \mapsto J_Y f(Z) = E_Z L_Y f(X)$ the Jacobi form. If $Y = Z$, then

$$\int_a^b L_Y L_Y f(X) dt = \frac{d^2 J}{d\epsilon^2}.$$

Assuming that M has a connection ∇ , we embark on the computation of $J_Y f(Z)$. Recall that

$$L_Y f = \nabla_Y f + \dot{f}(\nabla_X Y) + \dot{f}(T(Y, X))$$

and

$$E_Z h = \nabla_Z h - \nabla_X \dot{h}(Z) + \dot{h}(T(Z, X)),$$

for $h \in C^\infty(TM)$. (We have stopped writing out the additional X everywhere, e.g. writing $L_Y f$ for $L_Y f(X)$.)

Starting with $E_Z \nabla_Y f$ we get

$$(2.1) \quad E_Z \nabla_Y f = \nabla_Z \nabla_Y f - \nabla_X \dot{\nabla}_Y f(Z) + \dot{\nabla}_Y f(T(Z, X)).$$

With $h = \dot{f}(\nabla_X Y) + \dot{f}(T(Y, X))$, we have

$$\dot{h}(U) = \dot{f}(\nabla_U Y + T(Y, U)) + \ddot{f}(\nabla_X Y + T(Y, X), U),$$

where we have defined the second order fiber derivative $\ddot{f}$ in the obvious way. Thus

$$(2.2) \quad \dot{h}(T(Z,X)) = \dot{f}(\nabla_{T(Z,X)}Y + T(Y,T(Z,X))) + \ddot{f}(\nabla_X Y + T(Y,X), T(Z,X)).$$

We also need

$$(2.3) \quad \begin{aligned} \nabla_X \dot{h}(Z) &= \nabla_X \dot{f}(\nabla_Z Y + T(Y,Z)) + \\ &+ \dot{f}(\nabla^2 Y(;Z;X) + (\nabla_X T)(Y,Z) + T(\nabla_X Y, Z)) + \\ &+ \nabla_X \ddot{f}(\nabla_X Y + T(Y,X), Z) + \ddot{f}(\nabla_X^2 Y + \nabla_X T(Y,X), Z), \end{aligned}$$

$\nabla^2 Y(;Z;X)$ being the tensor $(X,Z) \rightarrow \nabla_X \nabla_Z Y - \nabla_{\nabla_X Z} Y$, and $\nabla_X \ddot{f}$ defined in the same fashion as $\nabla_X \dot{f}$. Finally, we have

$$(2.4) \quad \begin{aligned} \nabla_Z h &= Zh - \dot{h}(\nabla_Z X) = \nabla_Z \dot{f}(\nabla_X Y + T(Y,X)) + \\ &+ \dot{f}(\nabla^2 Y(;X;Z) + (\nabla_Z T)(Y,X) + T(\nabla_Z Y, X)) - \\ &- \ddot{f}(\nabla_X Y + T(Y,X), \nabla_Z X). \end{aligned}$$

Set $\nabla^2 f(;Y;Z) = \nabla_Z \nabla_Y f - \nabla_{\nabla_Z Y} f$. Then

$$(2.5) \quad \nabla_Z \nabla_Y f - \nabla_X \dot{f}(\nabla_Z Y) + \dot{f}(T(\nabla_Z Y, X)) = \nabla^2 f(;Y;Z) + \mathbb{E}_{\nabla_Z Y} f.$$

Also

$$(2.6) \quad \widehat{\nabla_Z \dot{f}(\nabla_X Y + T(Y,X))} = \nabla_Z \dot{f}(\nabla_X Y + T(Y,X)) - \ddot{f}(\nabla_Z X, \nabla_X Y + T(Y,X)),$$

and

$$(2.7) \quad \nabla^2 Y(;X;Z) - \nabla^2 Y(;Z;X) + \nabla_{T(Z,X)} Y = R(Z,X)Y.$$

Now

$$J_Y f(Z) = \mathbb{E}_Z \nabla_Y f + \nabla_Z h - \nabla_X \dot{h}(Z) + \dot{h}(T(Z,X)).$$

Assume that $E f = 0$. Then using the symmetry of $\ddot{f}$ and (2.1)-(2.7), we have

$$\begin{aligned}
 (2.8) \quad J_Y f(Z) &= \nabla^2 f(, Y; Z) - \nabla_X \dot{\nabla}_Y f(Z) + \dot{\nabla}_Y f(T(Z, X)) + \\
 &+ \dot{\nabla}_Z f(\nabla_X Y + T(Y, X)) + \dot{f}(R(Z, X)Y + (\nabla_Z T)(Y, X) - (\nabla_X T)(Y, Z) - \\
 &- T(\nabla_X Y, Z) + T(Y, T(Z, X))) - \nabla_X \dot{f}(T(Y, Z)) - \nabla_X \ddot{f}(\nabla_X Y + T(Y, X), Z) - \\
 &- \ddot{f}(\nabla_X^2 Y + \nabla_X T(Y, X), Z) + \ddot{f}(\nabla_X Y + T(Y, X), T(Z, X)).
 \end{aligned}$$

Hence $J_Y f(Z)$ is seen to depend on the values of X , Y , and Z along the curve only.

If $T = 0$, the formula reduces to

$$\begin{aligned}
 (2.9) \quad J_Y f(Z) &= \nabla^2 f(, Y; Z) - \nabla_X \dot{\nabla}_Y f(Z) + \dot{\nabla}_Z f(\nabla_X Y) + \dot{f}(R(Z, X)Y) - \\
 &- \nabla_X \ddot{f}(\nabla_X Y, Z) - \ddot{f}(\nabla_X^2 Y, Z).
 \end{aligned}$$

When in addition $\nabla f = 0$, the first three terms in (2.9) drop out.

Example

Assume that M is Riemannian with metric tensor g and associated Levi-Civita connection ∇ . Then $\nabla g = 0$ and $T = 0$. Let $f(X) = \frac{1}{2}g(X, X)$. In that case, we have

$$\left\{ \begin{array}{l} \nabla f = 0 \\ \dot{f}(U) = g(U, X) \\ \ddot{f}(U, V) = g(U, V) \\ \nabla \dot{f} = 0. \end{array} \right.$$

Thus

$$J_Y f(Z) = g(R(Z,X)Y,X) - g(\nabla_X^2 Y, Z)$$

and

$$J_Y f = 0 \iff \nabla_X^2 Y + R(Y,X)X = 0.$$

3. The second variation for multiple integrals

In this section we assume that f is a C^∞ function on the p -fold fiber product $TM \oplus \dots \oplus TM$ (p times), where $p \leq \dim M$. Suppose that we have given p commuting vector fields $X_1, \dots, X_p$, defined on the same open subset of M as a vector field Y . Finally, assume that M has a torsion-free connection ∇ .

Then proceeding as in section 1 (see [1] for the details) we obtain

$$L_Y f = \nabla_Y f + \dot{f}^j (\nabla_{X_j} Y)$$

and

$$E_Y f = \nabla_Y f - \nabla_{X_j} \dot{f}^j(Y),$$

where we for the sake of brevity have suppressed the variables $X_1, \dots, X_p$, having for example written $L_Y f$ instead of $L_Y f(X_1, \dots, X_p)$. Also, the following definitions, analogous to those in the one-variable case, ought to be pointed out:

$$\dot{f}^j(U) = \left. \frac{d}{d\epsilon} f(X_1, \dots, X_j + \epsilon U, \dots, X_p) \right|_{\epsilon=0},$$

$$\nabla_U f = Uf - \dot{f}^j (\nabla_U X_j),$$

and

$$\nabla_U \dot{f}^j(V) = U\dot{f}^j(V) - \dot{f}^j(\nabla_U V)$$

for vector fields U and V .

To apply this to a (local) variational problem, we let S be an integral surface of (linearly independent) $X_1, \dots, X_p$ with local

coordinates $\frac{\partial}{\partial x^i} = X_i$. If Y and Z are two vector fields along S , extend them in some way. Consider the integral

$$J = \int_S f(X_1, \dots, X_p) dx^1 \dots dx^p.$$

Two transformations and derivations give

$$\int_S L_Z L_Y f dx^1 \dots dx^p.$$

Since $L_Y f = Yf - \dot{f}^j([Y, X_j])$, we obtain

$$\begin{aligned} L_Z L_Y f &= ZYf - Z\dot{f}^j([Y, X_j]) - Y\dot{f}^k([Z, X_k]) + \\ &+ \dot{f}^j([Y, [Z, X_k]]) + \dot{f}^{jk}([Y, X_j], [Z, X_k]), \end{aligned}$$

where $\ddot{f}^{jk}$ is the second order partial fiber derivative defined by

$$\ddot{f}^{jk}(U, V) = \left. \frac{d}{ds} \frac{d}{dt} f(\dots, X_j + sU, \dots, X_k + tV, \dots) \right|_{\substack{s=0 \\ t=0}}.$$

It is easy to see that

$$\ddot{f}^{jk}(U, V) = \ddot{f}^{kj}(V, U).$$

Thus Jacobi's identity for vector fields yields

$$L_Z L_Y f - L_Y L_Z f = L_{[Z, Y]} f.$$

If $E f = 0$ on S and at least one of Y , Z , or f has compact support, then an integration by parts gives

$$\int_S L_Z L_Y f dx^1 \dots dx^p = \int_S L_Y L_Z f dx^1 \dots dx^p.$$

Defining the Jacobi form by $J_Y f(Z) = E_Z L_Y f$ and after two more integrations by parts, we find that

$$(3.1) \quad \int_S J_Y f(Z) \, dx^1 \dots dx^p = \int_S J_Z f(Y) \, dx^1 \dots dx^p.$$

Performing a calculation analogous to the one leading to (2.8), but with $T = 0$, we get

$$\begin{aligned} J_Y f(Z) = & \nabla^2 f(\cdot; Y; Z) - \nabla_{X_k} \nabla_Y \dot{f}^k(Z) + \dot{\nabla}_Z f^j(\nabla_{X_j} Y) + \dot{f}^j(R(Z, X_j)Y) - \\ & - \nabla_{X_k} \ddot{f}^{jk}(\nabla_{X_j} Y, Z) - \ddot{f}^{jk}(\nabla_{X_k} \nabla_{X_j} Y, Z). \end{aligned}$$

Here we have assumed that $E f = 0$. Observe that this expression depends upon the behavior on S only of the vector fields involved.

If f is a scalar density, then $L_Y f$ is too. As we have shown in Proposition 1.1 of [1], the Euler derivative of a scalar density is also a scalar density. Hence, using a partition of unity, we can extend (3.1) to general orientable manifolds S so long as f is a scalar density, and the various integrals are meaningful (if Y and Z have, or f has compact support, for instance).

We recall also Proposition 1.2 from [1] stating that $E_Y f = 0$ if Y is tangential to S and f is a scalar density. Set $Z = TZ + \perp Z$, separating Z into its tangential and normal components. Then, letting dx denote $dx^1 \dots dx^p$,

$$(3.2) \quad \int_S J_Y f(Z) \, dx = \int_S J_Y f(\perp Z) \, dx = \int_S J_{\perp Z} f(Y) \, dx = \int_S J_{\perp Y} f(\perp Z) \, dx.$$

under the conditions mentioned above needed to interchange the vector fields.

Example

Let M be a Riemannian manifold with metric tensor $g(\cdot, \cdot)$, and let S be a minimal surface in M . Consider the Lagrangian

$f = \sqrt{g}$, where $g = \det g(X_s, X_t)$. Then $\nabla f = 0$ and

$$J_Y f(Z) = \dot{f}^j(R(Z, X_j)Y) - \nabla_{X_k} \dot{f}^{jk}(\nabla_{X_j} Y, Z) - \dot{f}^{jk}(\nabla_{X_k} \nabla_{X_j} Y, Z).$$

Furthermore,

$$\left\{ \begin{aligned} \dot{f}^j(U) &= \sqrt{g} g^{js} g(U, X_s) \\ \dot{f}^{jk}(U, V) &= \sqrt{g} [g^{js} g(U, X_s) g^{kt} g(V, X_t) - g(U, X_s) g^{jk} g(V, X_t) g^{ls} - \\ &\quad - g(U, X_s) g^{jl} g(X_l, V) g^{ks} + g^{jk} g(U, V)] . \end{aligned} \right.$$

Before starting the calculation of the Jacobi form for f , we remark that we will use normal coordinates on S , as we may since $f = \sqrt{g}$ is a scalar density with respect to orientation-preserving coordinates.

First,

$$(3.3) \quad \dot{f}^j(R(Z, X_j)Y)/\sqrt{g} = g^{js} g(R(Z, X_j)Y, X_s) = R_S(Z, Y),$$

R_S denoting the "partial Ricci tensor with respect to S ".

Secondly, assuming that Z is normal,

$$\dot{f}^{jk}(\nabla_{X_k} \nabla_{X_j} Y, Z)/\sqrt{g} = g^{jk} g(\nabla_{X_k} \nabla_{X_j} Y, Z).$$

Now, if Y also is normal to S , then a separation of $\nabla_{X_j} Y$ into tangential and normal components gives

$$\nabla_{X_j} Y = -A_Y(X_j) + D_{X_j} Y.$$

We have used $g(\nabla_{X_s} X_t, Y) = g(A_Y(X_s), X_t)$ (which is symmetric in X_s and X_t) and the induced connection D on the normal bundle. Because

Z is normal,

$$g(\nabla_{X_k} \nabla_{X_j} Y, Z) = -g(A_Z(A_Y(X_j)), X_k) + g(D_{X_k} D_{X_j} Y, Z).$$

In normal coordinates $g^{jk} D_{X_k} D_{X_j} Y = \Delta_D Y$, where Δ_D denotes the Laplace operator on the normal bundle with respect to the connection D. Thus

$$(3.4) \quad \ddot{f}^{jk}(\nabla_{X_k} \nabla_{X_j} Y, Z) = \sqrt{g}(-\text{tr } A_Z \circ A_Y + g(\Delta_D Y, Z)).$$

The worst calculation comes last. It becomes easier, if we remember that we are using normal coordinates and that Z is normal. Then we have

$$\begin{aligned} \nabla_{X_k} \ddot{f}^{jk}(\nabla_{X_j} Y, Z) / \sqrt{g} &= g^{js} g(\nabla_{X_j} Y, X_s) g^{kt} g(Z, \nabla_{X_k} X_t) - \\ &- g(\nabla_{X_j} Y, X_s) g^{jk} g(Z, \nabla_{X_k} X_1) g^{1s} - g(\nabla_{X_j} Y, X_s) g^{jl} g(\nabla_{X_k} X_1, Z) g^{ks}. \end{aligned}$$

Our surface is minimal, so $g^{kt} \nabla_{X_k} X_t = 0$, and the first term vanishes.

We then have

$$\begin{aligned} \nabla_{X_k} \ddot{f}^{jk}(\nabla_{X_j} Y, Z) / \sqrt{g} &= g(A_Y(X_j), X_s) g^{jk} g(A_Z(X_1), X_k) g^{1s} + \\ &+ g(A_Y(X_j), X_s) g^{jl} g(A_Z(X_k), X_1) g^{ks}, \end{aligned}$$

whence

$$(3.5) \quad \nabla_{X_k} \ddot{f}^{jk}(\nabla_{X_j} Y, Z) = \sqrt{g}(2 \text{tr } A_Z \circ A_Y).$$

Combining (3.3), (3.4), and (3.5) we obtain for normal Y and Z

$$J_Y f(Z) = \sqrt{g}(R_S(Z, Y) - \text{tr } A_Z \circ A_Y - g(\Delta_D Y, Z)).$$

Hence, for the second variation of the volume V of a minimal surface S (with boundary) with respect to a variation with compact support (away from the boundary) with variation vector field Y , we obtain by (3.2)

$$\left. \frac{d^2 V}{d\epsilon^2} \right|_{\epsilon=0} = \int_S \left[R_S(\perp Y, \perp Y) - \text{tr } A_{\perp Y}^2 - g(\Delta_D \perp Y, \perp Y) \right] dS,$$

dS being the volume element on S .

References

- [1] Kremer, W.: A variational proof of the Gauss-Bonnet formula, Technical report, Lund, 1980.
- [2] Peetre, J.: A "non-linear" exterior derivative and the foundations of the calculus of variations, Technical report, Lund, 1976.
- [3] Peetre, J.: The Euler derivative, an intrinsic approach to the calculus of variations, Math. Scand. 42 (1978), 313-333.
- [4] Peetre, J.: Further comments on the Euler derivative, Technical report, Lund, 1977.
- [5] Spivak, M.: A comprehensive introduction to differential geometry, Volume IV. Publish or Perish, Inc., Boston, 1975.

